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GENERATION AND EVOLUTION OF OBLIQUE SOLITARY WAVES IN SUPERCRITICAL FLOWS

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Abstract – It is considered that a thin strut sits in a supercritical shallow water flow sheet over a homogeneous or very mildly varying topography. This stationary 3-D problem can be reduced from a Boussinesq-type equation into a KdV equation with a forcing term due to uneven topography, in which the transverse coordinate Y plays a same role as the time in original KdV equation. As the first example a multi-soliton wave pattern is shown by means of N-soliton solution. The second example deals with the generation of solitary wave-train by a wedge-shaped strut on an even bottom. Whitham's average method is applied to show that the shock wave jump at the wedge vertex develops to a cnoidal wave train and eventually to a solitary wavetrain. The third example is the evolution of a single oblique soliton over a periodically varying topography. The adiabatic perturbation result due to Karpman & Maslov (1978) is applied. Two coupled ordinary differential equations with periodic disturbance are obtained for the soliton amplitude and phase. Numerical solutions of these equations show chaotic patterns of this perturbed soliton. © Elsevier, Paris

1. Introduction

This paper contains a theoretic investigation of the problem of generation of oblique solitary waves by a fixed slender obstacle and their evolution over an even or uneven bottom. This problem has a particular bearing on, and was suggested by, the behaviour of ships moving at supercritical speeds in still water of restricted depth, and may be more directly related to a variety of problems involving shallow water, such as river flow past obstacles. The critical speed is known to be $\sqrt{g^* h_0^*}$, which is the speed of the wave with infinitely large wave length, and the depth Froude number U can be understood as the ratio of mean flow speed to critical speed. Supercritical flow means $U > 1$. For the sake of simplicity we study here only the case of a thin obstacle which is fixed on the bottom and extends with vertical sides all the way from free surface to the bottom. For definiteness we refer to the thin obstacle as a strut.

It is well known that the problem of the strut shallow water flow under the linear and nondispersive assumptions is entirely congruent to the aerodynamics of a 2-D thin wing. The depth Froude number plays the same role as the Mach number in the aerodynamics. The strut in a supercritical flow will generate oblique shock waves like a supersonic wing. The first solution appears to have been given by Michell in his famous wave-resistance paper (Michell 1898), not only for subcritical but also for supercritical cases. In the sixties and seventies the theory has been developed for practical purpose. Tuck (1966) presented a linear technique of matched asymptotic expansions for a slender ship in shallow water, where the dispersion effect was still excluded. Lea & Feldman (1972) partly took account of nonlinearity and used an established numerical method of transonic flow for computing the steady transcritical motion of ships. Later on Mei (1976) extended this

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work to include the dispersion effect in the transcritical range while dealing with the steady problem. One of his results is that a thin strut can generate an oblique solitary wave, if its shape is suitably chosen. In the meantime Karpman (1975, p. 92) studied the problem of flow around a 2-D thin body in a dispersive medium, especially for supersonic cases. More theoretic understandings of generation and evolution of solitons are achieved.

On the other side, the dynamic behaviour of solitons under perturbations becomes a hot topic. Two enlightening reviews in this field were given by Abdullaev (1989) and Kivshar & Malomed (1989). Most of works are based on the so-called inverse-scattering perturbation, of which a most protruding special case is the adiabatic approximation. Thereby the original problems of partial differential equations are reduced into finite-dimensional dynamic systems in terms of amplitude, phase and speed of solitons. The reduced systems reflect more or less the physical phenomena. As a concrete example, the author of this paper discovered theoretically as well as experimentally, see Chen & Wei (1994, 1996), that a nonlinear Schrödinger equation-type soliton does chaotic motions in time under perturbations. There is another type of chaos of solitons, namely, in space. From dynamic system's point of view, if a soliton, seen as a homoclinic orbit, is disturbed by some periodic effects, the chaos can occur. The reason is following. The stable and unstable manifolds of Poincaré maps stemming from the same saddle point can intersect at other point else, so that a Smale horseshoe is formed and, this means that, the set of Poincaré maps becomes a Cantor one. Melnikov's integral can judge whether both manifolds intersect. Kirchgässner (1991) showed by means of Melnikov's integral that if a capillary gravitational solitary wave is disturbed by a stationary periodic pressure, spatial chaotic wave patterns can be formed. One of reduced cases of the problem concerned in this paper can be also analysed in the same way.

In this paper we consider a supercritical shallow water flow past a thin strut, which is fixed on a homogeneous or very mildly varying topography. The strut generates oblique water waves that are similar to shock waves in supersonic flow but dissimilarly evolve eventually to solitary waves due to the balance of dispersive and nonlinear effects. We start with a stationary Boussinesq-type equation, namely a Kadomtsev-Petviashvili (KP) equation, and reduce it into a KdV equation, in which the transverse coordinate Y plays the same role as the time in a usual KdV equation. Three cases are investigated: generation of N-soliton wave pattern by a semi-infinite thin strut on an even bottom; generation of solitary wave-train by a wedge-shaped strut on the even bottom as well; and evolution of a single oblique soliton over a periodically varying topography. In the first case the well-known N-soliton solution of the KdV equation is used to show a multi-soliton wave pattern. In the second case Whitham's average method is applied to show that the shock wave jump at the wedge vertex will develop to a cnoidal wave train and eventually to a solitary wavetrain, which is the same solution as that of stepwise initial value problem of KdV equation studied by Zakharov et al. (1980). In the third case, i.e. an oblique soliton over an uneven topography, the adiabatic perturbation result due to Karpman & Maslov (1978) is applied. Two coupled ordinary differential equations with periodic disturbance are obtained for the soliton amplitude and phase. Numerical solutions of these equations show chaotic motions of this perturbed soliton. Although the reduced model, KdV equation, holds both for finite and infinite struts, the case calculations in this paper do not apply to the finite struts that generate trailing oscillating waves and appear in most physical problems.

2. Formulation

Let's consider a strut sits in a shallow water flow sheet over an inhomogeneous bottom and a constant incident stream U runs in the negative x-direction. The strut is relatively still to the bottom, so the problem can be simplified to be stationary. If the incident stream velocity U^* is larger than the critical value $\sqrt{g^* h_0^*}$, the wave pattern generated by a strut is similar to shock wave pattern by a thin aerofoil at a supersonic speed. But because there is the dispersion effect in shallow water, the wave pattern has its own nature. Dimensional variables are marked by asterisks "*".

The study begins with the shallow water model formulated in Chen & Sharma (1994, Eq. 19). Here the uneven bottom effect is taken into account additionally but only up to its first order, since the variation of bottom is assumed to be very mild. In terms of standard normalisation of variables by shallow water approximation,

$$(x,y)=\mu(x^*,y^*)/h_0^*,\quad z=z^*/h_0^*,\quad t=\mu t^*\sqrt{g^*h_0^*}/h_0^*,$$

$$\zeta=\frac{\zeta^*}{\varepsilon h_0^*},\quad \varphi=\frac{\mu\varphi^*}{\varepsilon h_0^*\sqrt{g^*h_0^*}},\quad p=\frac{p^*}{\rho^*g^*h_0^*},\quad h=h^*/h_0^*,\quad b=\mu b^*/(\varepsilon h_0^*), \tag{1}$$

the problem is governed by a Boussinesq-type equation, or called stationary extended Kadomtsev-Petviashvili (eKP) equation, in the field

$$(1-U^2)\varphi_{xx}+\varphi_{yy}+\varepsilon U\varphi_x\varphi_{yy}+\frac{\mu^2}{3}U^2\nabla^2\varphi_{xx}+3\varepsilon U\varphi_x\varphi_{xx}+2\varepsilon U\varphi_{xy}\varphi_y=\varepsilon U\frac{\partial h(x,y)}{\partial x} \tag{2}$$

and the boundary condition on the side walls of strut $y=\pm\varepsilon b(x)/2$,

$$\frac{\partial\varphi}{\partial y}(x,y=\pm 0)=\mp\frac{1}{2}U\frac{db(x)}{dx}, \tag{3}$$

where ε and μ are two smallness parameters that are defined as the ratios of typical wave amplitude to mean water depth and mean water depth to typical wave length, φ is depth-averaged velocity potential and $b(x)$ is the normalised width of strut. The elevation ζ of free surface and the pressure p are expressed approximately as

$$\zeta(x,y)=U\varphi_x+O(\mu^2,\varepsilon),\quad p(x,y,z)=\varepsilon\zeta-z+\frac{\mu^2}{2}\left[(1+\varepsilon\zeta)^2-(1+z)^2\right]\nabla^2\zeta+O(\varepsilon\mu^2). \tag{4}$$

Without loss of generality we set $\varepsilon=\mu^2$.

Because of symmetry of the problem, we need only to consider the half plane $y>0$. For supercritical speed $U>1$ and $\sqrt{U^2-1}=O(1)$, on the ground that the causal waves are downstream we can apply a characteristic transformation for the field $y>0$,

$$\xi=x+y\sqrt{U^2-1},\quad Y=\frac{\varepsilon y}{\sqrt{U^2-1}}. \tag{5}$$

Then via

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial\xi},\quad \frac{\partial}{\partial y}=\sqrt{U^2-1}\frac{\partial}{\partial\xi}+\frac{\varepsilon}{\sqrt{U^2-1}}\frac{\partial}{\partial Y},$$

(2) and (3) become

$$\varphi_{\xi Y}+\frac{3}{2}U^3\varphi_\xi\varphi_{\xi\xi}+\frac{1}{6}U^4\varphi_{\xi\xi\xi\xi}=\frac{U}{2}\frac{\partial h(\xi-y\sqrt{U^2-1},y)}{\partial\xi}, \tag{6}$$

and the boundary condition at $Y=+0$,

$$\varphi_\xi(\xi,Y=+0)=-\frac{U}{2\sqrt{U^2-1}}\frac{db(x)}{dx}. \tag{7}$$

It is clear that the problem is a standard initial-value problem of the KdV equation if the forcing term on the right-hand side is ignored and φ_ξ is seen as the unknown variable. The variable Y plays the same role as time in the original KdV equation. The problem was extensively investigated. For some specially localised initial profiles, from which multiple solitons evolve, it can be solved in closed form by the well-known inverse scattering method. For more general initial profile or perturbed KdV problems it can be analysed by various asymptotic methods. For instance, two well-known ones are Whitham's average method (1965, 1967, 1974) for wavetrains and the adiabatic method, e.g. see Abdullaev (1989) and Kivshar & Malomed (1989), for perturbed solitons.

This reduction approach was first applied in a similar model equation for the problem of body flow in a dispersive medium by Karpman (1975, § 22). It shows here that all solutions of the KdV equation are approximate solutions of the stationary extended KP equation. Since the KdV equation is easy to solve, this reduction is very useful in the context of this paper. Despite easiness of its numerical solution, the problem of closed struts of finite extent can be thereby analysed asymptotically, see Karpman (1975) and Drazin & Johnson (1989). The wave pattern generated by a finite strut can be easily imagined with help of a solution of initial value problem of KdV equation, whose initial profile looks like a lying "S". For instance, the numerical result shown in Fig. 21.3 of Karpman (1975, p.93) can be interpreted that the forebody generates multiple solitons, meanwhile the afterbody makes an oscillating wave train behind. Both soliton and oscillating wave trains extend along the characteristic lines obliquely downstream to infinity.

The solutions in this paper are restricted to concern only solitary waves. Thus the struts considered should be of infinite extent. In the following three sections the above mentioned methods are applied and three different kinds of 3-D wave patterns generated by struts are demonstrated. The first two cases deal with even bottom and the last case concerns uneven bottom.

3. Wave pattern of N-soliton solution

The strut can be so chosen that it generates single soliton or multi solitons. To show this phenomenon we directly employ results given in the book by Drazin & Johnson (1989). Via a transformation

$$u = \frac{U^3}{4}\varphi_\xi, \quad T = \frac{\sqrt{6}}{U^2}Y, \quad X = \frac{\sqrt{6}}{U^2}\xi, \tag{8}$$

(6) becomes the standard form of KdV equation, which in spite of the forcing term f, differs from (4.7) in Drazin & Johnson (1989) only in a minus in front of the nonlinear term,

$$u_T + 6uu_X + u_{XXX} = f, \qquad f = \frac{U^4}{8}\frac{\partial h(X,T)}{\partial X} \tag{9}$$

with the "initial condition" due to (7)

$$u(X, T=0) = -\frac{U^4}{8\sqrt{U^2-1}}\frac{db(x)}{dx}. \tag{10}$$

So now if $h(X,T) = 1$, i.e. $f = 0$, and the "initial condition" is taken as

$$\varphi_\xi(\xi, 0) = \frac{4N(N+1)}{U^3}\text{sech}^2\frac{\sqrt{6}}{U^2}(\xi - x_0), \tag{11}$$

i.e. the strut is of the form

$$\frac{db(x)}{dx} = -\frac{8N(N+1)\sqrt{U^2-1}}{U^4}\text{sech}^2\frac{\sqrt{6}}{U^2}(x - x_0), \tag{12}$$

then N solitons will evolve asymptotically from this initial single-peak wave packet. For $N = 1$ we have single soliton solution

$$\varphi_\xi(\xi, Y) = \frac{8}{U^3}\text{sech}^2\frac{\sqrt{6}}{U^2}(\xi - 4Y - x_0). \tag{13}$$

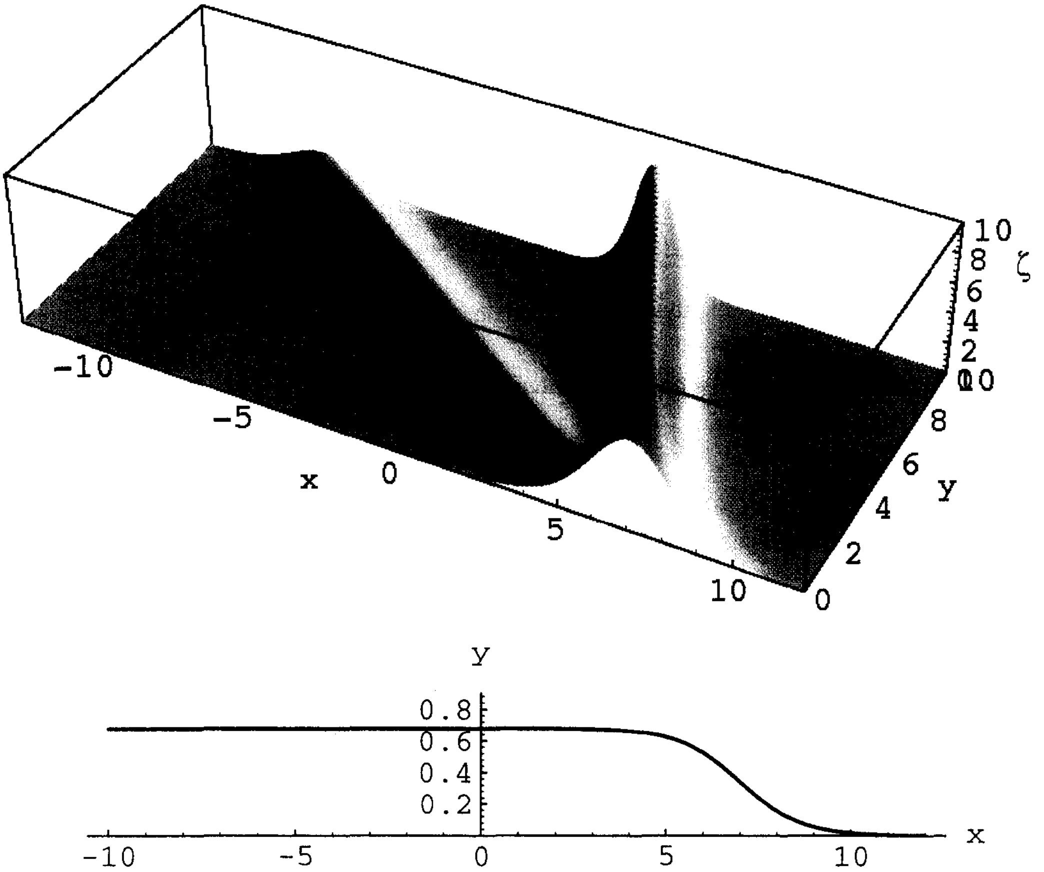


FIGURE 1. Strut shape and its wave pattern resulting from the double-soliton solution as $\varepsilon = 0.08$, $x_0 = 7$, $U = 2$ and $N = 2$.

This solution was first found by Mei (1976) for supercritical strut flow problem and called oblique solitary wave. More general for $N = 2$, from (4.36) in Drazin & Johnson (1989), we have the double-soliton solution

$$\varphi_\xi(\xi, Y) = \frac{48}{U^3}\frac{3 + 4\cosh[2(\xi - x_0) - 8Y]\sqrt{6}/U^2 + \cosh[4(\xi - x_0) - 64Y]\sqrt{6}/U^2}{\left[3\cosh(\xi - x_0 - 28Y)\sqrt{6}/U^2 + \cosh(3(\xi - x_0) - 36Y)\sqrt{6}/U^2\right]^2}. \tag{14}$$

It is asymptotic wave form for $Y \to +\infty$ can be given as

$$\varphi_\xi(\xi, Y) = \frac{32}{U^3}\mathrm{sech}^2\frac{\sqrt{6}}{U^2}\left[2(\xi - x_0 - 16Y) - \frac{1}{2}\log 3\right] + \frac{8}{U^3}\mathrm{sech}^2\frac{\sqrt{6}}{U^2}\left[(\xi - x_0 - 4Y) + \frac{1}{2}\log 3\right]. \tag{15}$$

The N-soliton solution belongs to so-called reflectionless initial profiles. For more general choice of $u(x, 0)$, one may not solve it in closed form, but one can easily solve it numerically or analyse it asymptotically, see Drazin & Johnson (1989, p. 81). Here we selected parameter values $\varepsilon = 0.08$, $x_0 = 7$, $U = 2$ and $N = 2$ and display the strut shape (12) and the wave pattern $\zeta = U\varphi_\xi$ of the double-soliton solution (14) in true perspective in Figure 1.

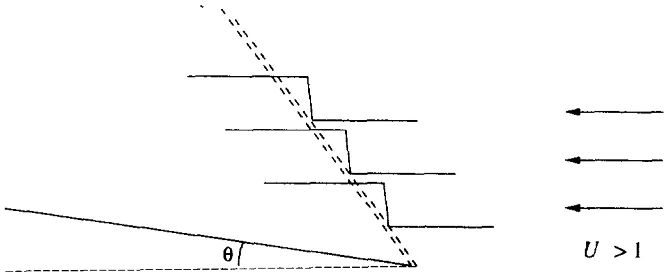


FIGURE 2. Oblique shock wave generated by a wedge-shape strut

Since the KdV equation is invariant under the transformation

$$u \to Au, \quad X \to X/\sqrt{A}, \quad T \to T/(A\sqrt{A}),$$

its solutions can be multiplied by an arbitrary positive factor A and correspondingly the variables X and T are divided by $\sqrt{A}$ and $A\sqrt{A}$. This means, the higher (lower) the solitary waves become, the narrower (wider) they are.

4. Wave pattern generated by a wedge-shaped strut

Now we consider a wedge-shaped strut on an even bottom. Let's first look at the simplest case, where both effects of nonlinearity and dispersion are neglected. Therefore the governing equation (2) becomes the wave equation for the case of $U > 1$,

$$(1 - U^2)\varphi_{xx} + \varphi_{yy} = 0, \tag{16}$$

and for the symmetric wedge-shaped strut flow the boundary condition (3) becomes

$$\frac{\partial \varphi}{\partial y}(x, y = \pm 0) = \begin{cases} 0, & x > 0, \\ \mp\frac{1}{2}U\frac{db(x)}{dx} = \mp const., & x \leq 0. \end{cases} \tag{17}$$

(16) has characteristic solutions $\varphi(x + y\sqrt{U^2 - 1})$ and $\varphi(x - y\sqrt{U^2 - 1})$. The physical solution is determined by the boundary condition, under consideration of the causality effect. For $y \geq 0$, it is

$$\frac{\partial \varphi}{\partial x}(x, y) = \frac{\partial \varphi}{\partial y}(x, y)/\sqrt{U^2 - 1} = \begin{cases} 0, & x + y\sqrt{U^2 - 1} > 0, \\ const., & x + y\sqrt{U^2 - 1} \leq 0. \end{cases} \tag{18}$$

This solution is schematically shown in Figure 2. That is a typical oblique shock wave solution in 2-D supersonic aerofoil flows. This stepwise shock wave stems from the wedge vertex and keeps permanent along the characteristic line.

The question is now how this shock wave evolves if the nonlinear and dispersive effects are taken into account. Since now the supercritical problem is already reduced into the initial value problem of KdV equation, it is possible to solve this problem of wedge-shaped strut in closed form.

4.1. *Whitham's average method*

The normal velocity on the strut sidewall is approximately constant. This means the "initial condition" (7) is not localised but constant in the semi-infinite line. Therefore the inverse scattering method is not available for this problem. Nevertheless we can use Whitham's average method to solve this problem.

Generally Whitham (1974) proposed the average variational approach for evolution of nonlinear dispersive wavetrains. As an example applied to KdV equation, he derived the modulation equations for the cnoidal wavetrain. The modulation equations are hyperbolic and there are three invariants along the characteristic lines. A more direct derivation was suggested by Manakov (1978) (see Zakharov et al 1980), that substituting the cnoidal wave solution into the KdV equation, while the parameters seen as functions of slow time and space, and integrating the equation over the periods of time and space will yield same modulation equations. The method was called direct perturbation method by some people and also corresponds to the general sense of "adiabatic approximation". In the following we state briefly the result of Whitham's method and the solution of stepwise initial value problem of KdV equation due to Zakharov et al (1980).

The problem concerned here becomes the standard form of KdV equation (9) with the stepwise initial condition

$$u(X,0) = \begin{cases} 0, & X > 0, \\ 1, & X \leq 0. \end{cases} \tag{19}$$

The constant initial value u_0 has been given by unit without loss of generality, because the smallness parameter ε is still free and can be evaluated correspondingly later on.

4.1.1. *Cnoidal Waves*

We are looking for the solution in the form of

$$u(X,T) = u(\vartheta), \quad \vartheta = X - CT. \tag{20}$$

Then (9) becomes

$$-Cu_\vartheta + 6uu_\vartheta + u_{\vartheta\vartheta\vartheta} = 0. \tag{21}$$

There are two immediate integrals

$$-Cu + 3u^2 + u_{\vartheta\vartheta} + B = 0,$$

$$-Cu^2 + 2u^3 + u_\vartheta^2 + 2Bu - 2A = 0.$$

We rewrite the last equation and define a cubic polynomial P_3 as

$$-\frac{1}{2}u_\vartheta^2 = u^3 - \frac{C}{2}u^2 + Bu - A := P_3(u). \tag{22}$$

The zeros p, q, r of the cubic polynomial P_3, shown in Figure 3, i.e.,

$$P_3(u) = (u-p)(u-q)(u-r), \quad p \geq q \geq r, \tag{23}$$

can be used as new variables in place of A, B, U. The relations between the coefficients and roots of the cubic polynomial are,

$$p + q + r = \frac{C}{2}, \quad pq + pr + qr = B, \quad pqr = A. \tag{24}$$

Since the left-hand side of (22) is negative, the right-hand side must be negative too for a real solution u. So the solution u must lie between the two zeros p and q which correspond to the heights of the crest and the trough, respectively.

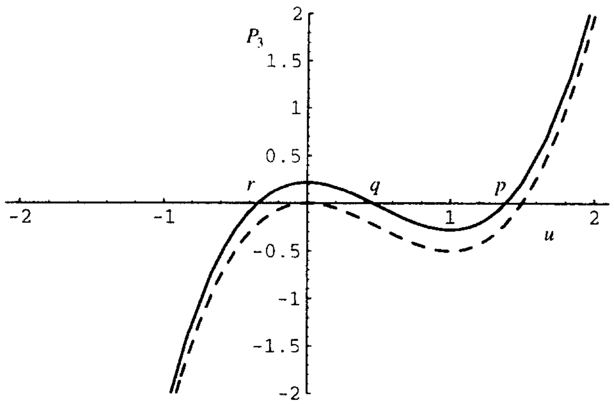


FIGURE 3. Roots of the cubic polynomial P_3.

Equation (22) can be integrated in terms of an elliptic integral by introducing

$$u = p\cos^2\psi + q\sin^2\psi, \quad \psi = \psi(\vartheta). \tag{25}$$

We insert (25) into (22). The left hand side becomes

$$-\frac{1}{2}u_\vartheta^2 = -2(p-q)^2\psi_\vartheta^2\cos^2\psi\sin^2\psi, \tag{26}$$

and the right hand side becomes

$$P_3 = -(p-q)^2\cos^2\psi\sin^2\psi\,[p-r-(p-q)\sin^2\psi]. \tag{27}$$

Thus we have

$$\psi_\vartheta^2 = \frac{1}{2}(p-r)[1-m\sin^2\psi], \quad m := \frac{p-q}{p-r}. \tag{28}$$

Integrating the above equation, we get and define

$$F(\psi, m) := \int_0^\psi \frac{d\psi}{\sqrt{1-m\sin^2\psi}} = \pm\frac{1}{\sqrt{2}}\sqrt{p-r}\,\vartheta, \tag{29}$$

where F is the incomplete elliptic integral of the first kind. The above relation can be regarded as an implicit equation for ψ as a function of ϑ, and inversely it defines

$$\cos\psi = \mathrm{Cn}\,[\frac{1}{\sqrt{2}}\sqrt{p-r}\,\vartheta, m], \quad \sin\psi = \mathrm{Sn}\,[\frac{1}{\sqrt{2}}\sqrt{p-r}\,\vartheta, m], \tag{30}$$

where Cn and Sn are the Jacobi elliptic cosine and sine functions. From Eq. (25) we have the expression

$$u = q + (p-q)\mathrm{Cn}^2[\frac{1}{\sqrt{2}}\sqrt{p-r}\,\vartheta, m]. \tag{31}$$

At this point it is useful to express the quantities concerned in terms of elliptic integrals. We introduce three variables, wave amplitude a, modulus m of elliptic integrals and average elevation β, where m is defined in (28) and a and β are defined as

$$a := \frac{p-q}{2}, \tag{32}$$

$$\beta := \bar{u} = p - 2a\frac{D(m)}{K(m)}. \tag{33}$$

where

$$D(m) = \frac{K(m) - E(m)}{m}$$

and $K(m)$ and $E(m)$ are the complete elliptic integrals of first and second kinds, respectively. If we rather use a, m and β as basic variables, we have

$$p = \beta + 2a\frac{D}{K}, \quad q = \beta + 2a\left(\frac{D}{K} - 1\right), \quad r = \beta + 2a\left(\frac{D}{K} - \frac{1}{m}\right), \tag{34}$$

and from (31)

$$u = q + 2a\mathrm{Cn}^2[\sqrt{\frac{a}{m}}\vartheta, m] = p - 2a\mathrm{Sn}^2[\sqrt{\frac{a}{m}}\vartheta, m] = r + \frac{2a}{m}\mathrm{Dn}^2[\sqrt{\frac{a}{m}}\theta, m]. \tag{35}$$

Since $\cos\psi$ is periodic in 2π, then $\mathrm{Cn}(z,m)$ is periodic in $4K(m)$ and $\mathrm{Cn}^2(z,m)$ in $2K(m)$. The wave number (2π/wavelength) and phase velocity are given by

$$k = \frac{\pi\sqrt{a}}{\sqrt{m}K}, \tag{36}$$

$$C = 2(p+q+r) = 6\beta + 4a\left(\frac{3D}{K} - \frac{1+m}{m}\right). \tag{37}$$

4.1.2. *Modulation Equations*

In the theory developed by Whitham (1965, 1967, see his book of 1974), it is assumed that the solution is given locally by the uniform solution (31) or (35) but that parameters p, q, r, or a, m, β, are now slowly varying functions of X and T. Then partial differential equations can be obtained for these functions by an appropriate averaging of the original equation. The motivation of the whole approach is discussed in great detail in Whitham (1965) and the averaged equations are obtained in various examples. However, the procedure can be simplified and given deeper significance by the so-called ***averaged variational principle*** cf. Whitham (1974).

Here we just rewrite the equations given by Whitham (1974, pp.565–570)

$$(q+r)_T + V_P(q+r)_X = 0, \tag{38}$$

$$(p+r)_T + V_Q(p+r)_X = 0, \tag{39}$$

$$(p+q)_T + V_R(p+q)_X = 0. \tag{40}$$

The three Riemann invariants are

$$P = q + r, \quad Q = p + r, \quad R = p + q, \tag{41}$$

the three characteristic lines, given names by P, Q, R, are

$$\frac{dX_P}{dT} = V_P, \quad \frac{dX_Q}{dT} = V_Q, \quad \frac{dX_R}{dT} = V_R, \tag{42}$$

where

$$V_P = C - \frac{4aK}{mD} = C - \frac{4aK}{K - E}, \tag{43}$$

$$V_Q = C - \frac{4a(1-m)K}{m(K-D)} = C - \frac{4a(1-m)K}{E - (1-m)K}, \tag{44}$$

$$V_R = C - \frac{4a(1-m)K}{m(mD-K)} = C - \frac{4a(1-m)K}{mE}. \tag{45}$$

In general the velocities V_P, V_Q, V_R are distinct and $V_P < V_Q < V_R$. Thus the system is hyperbolic. The limits $m \to 0$ and $m \to 1$ are both singular in that two of the velocities become equal. The limiting equations are then not strictly hyperbolic, although, due to the uncoupling of one of the equations, they may still be solved by integration along characteristics.

4.2. *Simple-wave solution*

By means of Whitham's equations, Zakharov, *et al* (1980) studied the stepwise initial value problem of KdV equation and obtained its asymptotic solution. The main results are stated here in a new way.

If a Riemann invariant is varying and others are constant throughout, the solution is enormously simplified. Such solution is called *simple wave* solution. In some sense it is a kind of self-similar solution. In this problem there is a simple-wave solution that corresponds to the asymptotic state of evolution of the initial stepwise profile, for which we are interested. We will see that one of Riemann invariants Q can vary from P to R. Therefore it is naturally believed that the oscillating range will be bounded on one side by $X^-(T)$, where $Q = P$, the amplitude a and modulus m are zero, and on the other side by $X^+(T)$, where $Q = R$ and consequently $m = 1$. Since $X^+(T) > X^-(T)$, we call $X^+(T)$ and $X^-(T)$ leading and trailing fronts, respectively.

Setting

$$\tau = X/T$$

and seeing P, Q and R only as functions of τ, the modulation equations (38)-(40) then become

$$\frac{d(P,Q,R)}{d\tau} \cdot [(V_P, V_Q, V_R) - \tau] = 0. \tag{46}$$

By virtue of simple-wave solution, we reasonably choose

$$P = const, \quad R = const, \text{ but } Q \neq const. \tag{47}$$

Then from (46) for Q we have

$$V_Q = \tau. \tag{48}$$

In terms of τ, following boundary conditions should be satisfied. On the leading front $\tau = \tau^+$,

$$m(\tau^+) = 1, \text{ i.e. } Q = R \tag{49}$$

and due to continuity of u,

$$u(\tau^+) = 0, \quad \text{or } \beta(\tau^+) = 0. \tag{50}$$

On the trailing front $\tau = \tau^-$,

$$a(\tau^-) = 0, \text{ i.e. } Q = P \tag{51}$$

and due to continuity of u,

$$u(\tau^-) = 1, \quad \text{or } \beta(\tau^-) = 1. \tag{52}$$

It is clear that there are no oscillating waves before the leading front and after the trailing front, i.e.

$$p = q = 0, \text{ as } \tau > \tau^+, \tag{53}$$

and

$$a(\tau) = 0, \quad u(\tau) = 1, \text{ as } \tau < \tau^-. \tag{54}$$

The leading-front conditions (49), (50), (53) yield, based on the fact that $D(m)/K(m) - 1 \to 0$ as $m \to 1$,

$$q = r, \quad 0 = \beta(\tau = \tau^+) = q = P/2,$$

i.e.

$$P = 0. \tag{55}$$

The trailing-front conditions (51), (52), (54) yield

$$p = q, \quad 1 = \beta(\tau = \tau^-) = p = R/2,$$

i.e.

$$R = 2. \tag{56}$$

From (34) we get

$$P = R - \frac{2a}{m} = 0, \quad Q = P + 2a,$$

i.e.

$$a = m, \quad Q = 2m. \tag{57}$$

C and β are evaluated as

$$C = P + Q + R = 2 + 2m, \quad \beta = 1 + m - 2mD(m)/K(m) = m - 1 + 2E(m)/K(m). \tag{58}$$

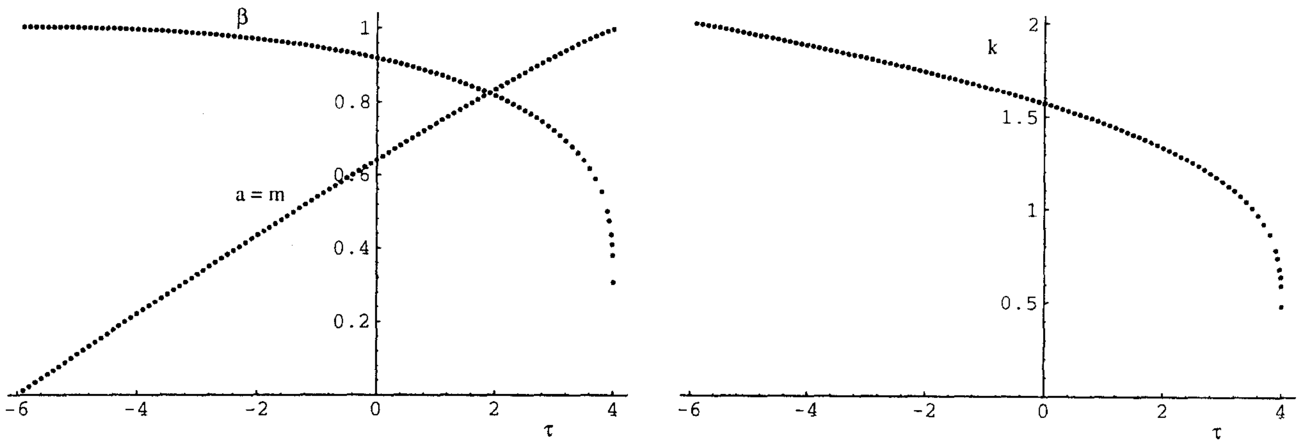


FIGURE 4. The wave amplitude $a = m$, modulus m, average elevation β and wave number k as functions of τ.

From (55), (56), (57), we get surprisingly simple expressions for p, q and r,

$$p = 1 + m, \quad q = 1 - m, \quad r = m - 1. \tag{59}$$

They can be also obtained by substituting expressions for a and β in (57) and (58) into (34), where all Jacobi elliptic functions disappear. Finally Eq. (48) with the expression (44) gives an algebraic equation for $m(\tau)$,

$$2(1+m) - \frac{4m(1-m)K(m)}{E(m) - (1-m)K(m)} = \tau. \tag{60}$$

Thus (58), (59), (60) and (31) provide the solution of the problem (9) with $f = 0$ and (19). The equation (60) is solved numerically, where $m = 1$ and $m = 0$ determine the leading and trailing fronts, that are $\tau^+ = 4$ and $\tau^- = -6$. For $\tau < \tau^-$ or $\tau > \tau^+$, we take $m = 0$ or $m = 1$, respectively. Figure 4 shows $a = m$, β and k as functions of τ between τ^- and τ^+.

The nature of waves near the leading and trailing fronts can be nevertheless ascertained analytically. Approximate formulas are listed below for $K(m)$ and $E(m)$ as $m \to 0$ and $m \to 1$. For m being small,

$$K(m) \approx \frac{\pi}{2}(1 + \frac{m}{4} + \frac{9}{64}m^2 + \cdots),$$

$$E(m) \approx \frac{\pi}{2}(1 - \frac{m}{4} - \frac{3}{64}m^2 + \cdots),$$

$$K(m) - E(m) \approx \frac{\pi}{2}(\frac{m}{2} + \frac{3}{16}m^2 + \cdots).$$

For $1 - m > 0$ being small

$$K(m) \approx \frac{1}{2}\ln\frac{16}{1-m},$$

$$E(m) \approx 1 + \frac{1}{4}(1-m)\left(\ln\frac{16}{1-m} - 1\right).$$

Near the trailing front $m \to 0$, (60) yields

$$-6 + 9m \approx \tau.$$

This gives

$$\tau^- = -6, \text{ and } \tau - \tau^- = 9m = 9a,$$

and means the amplitude a tends to zero linearly with respect to $\tau - \tau^-$.

Near the leading front $m \to 1$, (60) yields

$$4 - \frac{1}{2}(1-m)\ln\frac{16}{1-m} \approx \tau$$

This means

$$\tau^+ = 4, \text{ and } \tau^+ - \tau = \frac{1}{2}(1-m)\ln\frac{16}{1-m}.$$

It can be written approximately as $\tau^+ - \tau > 0$ but small,

$$1 - m \approx \frac{2(\tau^+ - \tau)}{\ln 1/(\tau^+ - \tau)}.$$

Therefore (36) yields that the wave number tends to zero as $m \to 1$ according to following rule,

$$k \approx \frac{2\pi}{\ln 1/(\tau^+ - \tau)}$$

and the second equation in (58) gives the local average elevation $\beta = \bar{u}$:

$$\beta \approx \frac{4}{\ln 1/(\tau^+ - \tau)}.$$

When $\tau = \tau^+$, $\beta(\tau)$ itself is zero and continuous at this point, but its derivative is infinitely large and therefore discontinuous at this point. We say there is a *weak* discontinuity on the leading front.

In order to express the solution in terms of true physical coordinates, we fall back on the transformations (5) and (8), and then we have

$$\tau = \frac{X}{T} = \frac{\xi}{Y} = \frac{x + y\sqrt{U^2-1}}{\varepsilon y/\sqrt{U^2-1}}.$$

This gives

$$\frac{x}{y} = \frac{\varepsilon\tau}{\sqrt{U^2-1}} - \sqrt{U^2-1}. \tag{61}$$

The oscillating wave train is bounded by the leading and trailing fronts, which are expressed in terms of true spatial coordinates as

$$\frac{x^\pm}{y} = \frac{\varepsilon\tau^\pm}{\sqrt{U^2-1}} - \sqrt{U^2-1}, \tag{62}$$

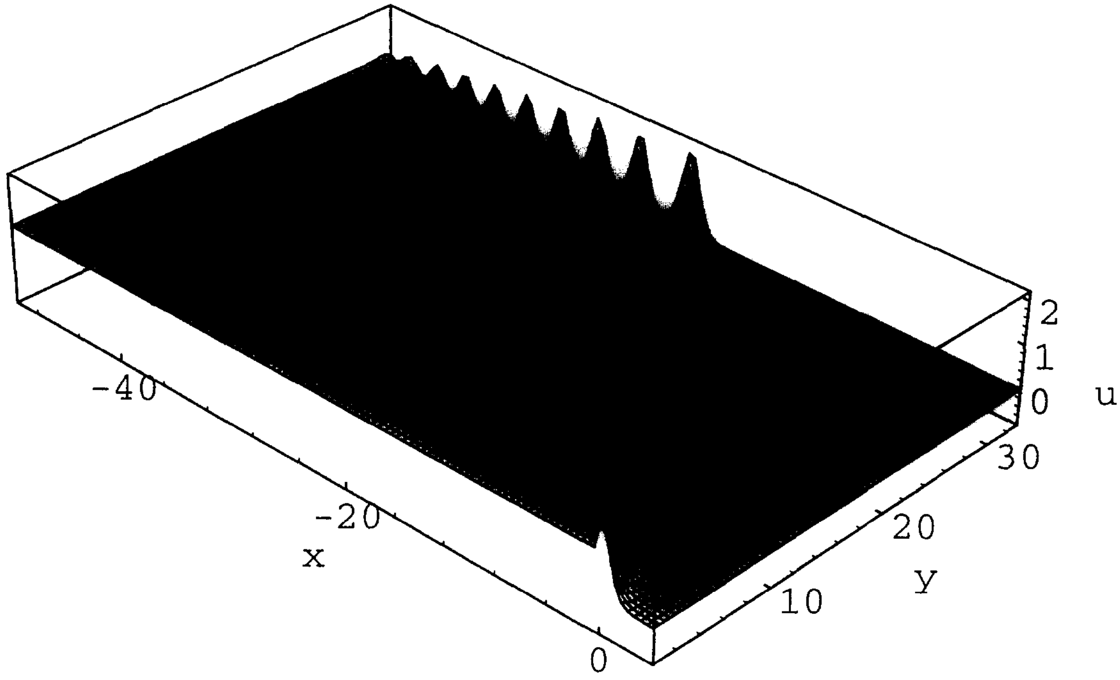


FIGURE 5. 3-D wave generated by a wedge-shaped strut in a supercritical stream in the case of $U = \sqrt{2}$ and $\varepsilon = 0.1$.

where $\tau^+ = 4$ and $\tau^- = -6$.

The half vertex angle θ of the wedge-shaped strut is related with the small parameter ε, if the initial step of u is unit. From (3) and (10) we have

$$\tan\theta = \frac{dy}{dx} = \frac{\varepsilon}{2}\frac{db}{dx} = \varepsilon\frac{8\sqrt{U^2-1}}{U^2}. \tag{63}$$

The approximate wave elevation $\zeta = U\varphi_x$ is associated with u as

$$\zeta = \frac{4}{U^3}u. \tag{64}$$

For $U = \sqrt{2}$ and $\varepsilon = 0.1$, then $\tan\theta = 0.2$ and $\zeta = \sqrt{2}u$.

The wave pattern is evaluated and demonstrated by the software Mathematica. Figure 5 shows the 3-D wave pattern generated by the wedge-shaped strut in true perspective for $U = \sqrt{2}$ and $\varepsilon = 0.1$. Two wave profiles at $y = 3$ and $y = 30$ are shown in Figure 6. As seen clearly a solitary wavetrain evolves from the initial stepwise profile, which is confined within the two characteristics τ^- and τ^+. Ahead of τ^+ there is no disturbance due to causality, while behind τ^- there is a constant disturbance, namely, a plateau. The reason for the plateau is the constant slope of wedge. The solution is believed to be true for the asymptotic state of large y, but questionable for small y. This is because of the limitation of the average method, by which only slow modulation can be taken account for.

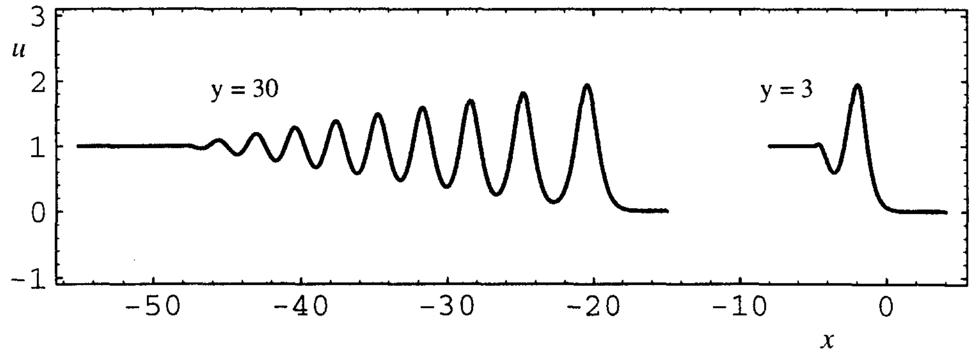


FIGURE 6. Wave profiles at $y = 3$ and 30.

5. Single oblique solitary wave perturbed by topography

In this section the evolution of a single oblique soliton over an uneven topography is studied. The single soliton solution of homogeneous KdV equation of (9) is in the form of

$$u = 2\kappa^2 \operatorname{sech}^2 z, \quad z = \kappa(X - \chi). \tag{65}$$

This soliton is generated by a suitably chosen thin strut like in Section 3. The perturbation due to the topography is assumed to be so small that the single soliton form keeps approximately unchanged. According to the adiabatic approximation due to Karpman & Maslov (1978), if the KdV equation is perturbed by a small forcing term f in (9), the amplitude κ and phase χ of the single soliton are governed by averaged equations

$$\kappa_T = -\frac{1}{4\kappa}\int_{-\infty}^{\infty} dz f \operatorname{sech}^2 z, \tag{66}$$

$$\chi_T = 4\kappa^2 - \frac{1}{4\kappa^3}\int_{-\infty}^{\infty} dz f \operatorname{sech}^2 z(z + \frac{1}{2}\operatorname{sech} 2z). \tag{67}$$

5.1. *ODEs for κ and χ*

The topography is assumed to be varying periodically, i.e. the forcing term is in the form of

$$f = A \sin\sigma X \cos\nu T.$$

Then the integrals in (66) and (67) can be carried out by using following results of definite integrals:

$$\int_{-\infty}^{\infty} \frac{\sin bz}{\sinh cz} dz = \frac{\pi}{c}\tanh\frac{b\pi}{2c}, \qquad Rec > |Imb|$$

$$\int_{-\infty}^{\infty} \frac{\cos bz}{\cosh cz} dz = \frac{\pi}{c}\operatorname{sech}\frac{b\pi}{2c}, \qquad Rec > |Imb|$$

$$\int_{-\infty}^{\infty} z \operatorname{sech}^2 z \sin(cz) dz = \frac{\pi}{2}\left[\pi c \coth(\pi c/2) - 2\right]$$

from Prudnikov et al. 1986 p. 467–668, and

$$\int_{-\infty}^{\infty} \operatorname{sech}^2 z \cos cz dz = \frac{\pi c}{\sinh(\pi c/2)}$$

from Holmes (1980). After straightforward manipulations we get two nonlinear ordinary differential equations for κ and χ,

$$\kappa_T = -\frac{\sigma\pi A \cos\nu T \sin\sigma\chi}{4\kappa^2} \operatorname{csch}\left(\sigma\pi/(2\kappa)\right), \tag{68}$$

$$\chi_T = 4\kappa^2 - \frac{A\cos\nu T}{4\kappa^3}\left[\cos\sigma\chi I_1(\sigma/\kappa) + \frac{1}{2}\sin\sigma\chi I_2(\sigma/\kappa)\right], \tag{69}$$

where

$$I_1(c = \sigma/\kappa) := \int_{-\infty}^{\infty} \sin(cz)\, z \operatorname{sech}^2 z dz = \frac{\pi}{2}\left[c\pi \coth\frac{c\pi}{2} - 2\right]$$

$$I_2(c = \sigma/\kappa) := \int_{-\infty}^{\infty} \cos cz \operatorname{sech} 2z \operatorname{sech}^2 z\, dz = \frac{c\pi}{2}\tanh\frac{c\pi}{4} - \frac{c\pi}{2}\tanh\frac{c\pi}{8} + \pi \operatorname{sech}\frac{c\pi}{4} - \frac{c\pi}{2}\operatorname{csch}\frac{c\pi}{4}.$$

5.2. *Numerical solutions*

We choose an example of $\sigma = 1$, $\nu = 0.6$, $A = 0.1$ and solve the ODEs (68) and (69) numerically by Mathematica. Figure 7 shows orbits in the phase plane $(\kappa, \dot{\kappa})$ in the ranges of $T = 0 - 150$ and $T = 100 - 120$. Figure 8 shows the soliton amplitude and phase (position) versus T.

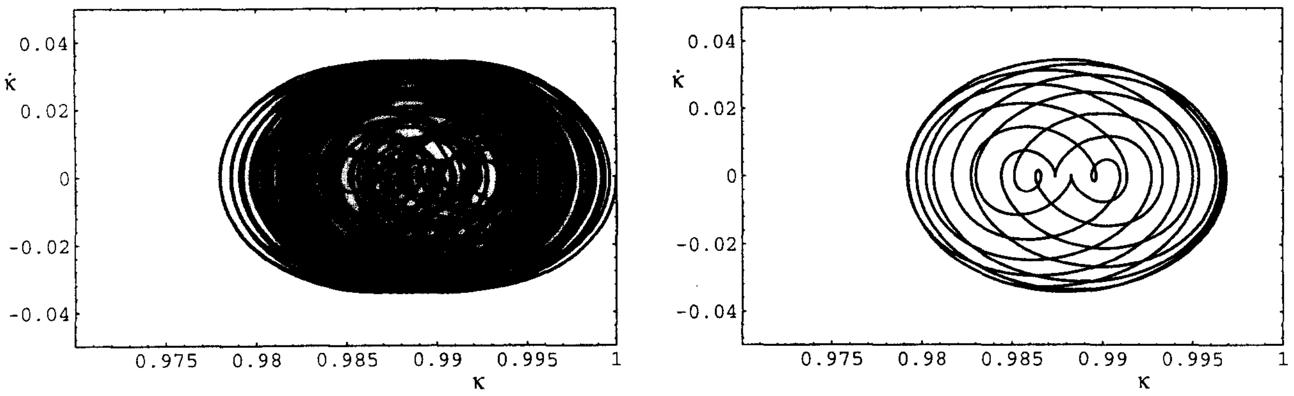
The perturbation of the so chosen topography is very small. The amplitude κ varies from 1 to 0.975, i.e. within 2.5 %, see two upper plots in Figure 8, and the perturbation effect on the phase χ is so small that it cannot be recognised in the plot of χ vs. T in the left-lower part of Figure 8. In order to see the effect we have to draw a plot of $\Delta\chi(T) = \chi(T) - \bar{\chi}(T)$ in a large magnifying power, shown in the right-lower part of Figure 8, where $\bar{\chi}$ is a linear fitting function, which passes through two points of $(100, \chi(100))$ and $(120, \chi(120))$.

It is seen clearly from Figures 7 and 8 that the soliton does chaotic motions in such a way that it varies in amplitude and phase. The physical scene is that a strut generates a solitary wave that extends obliquely into infinity and varies in amplitude and peak-position irregularly. Because of limitation of the adiabatic approximation, neither the waves generated by the uneven topography nor their effect on the soliton has been taken into account.

It is worth notice that one can show the existence of chaos near a homoclinic orbit analytically, if the reduced case is an ODE problem. Kirchgässner (1991) showed by means of Melnikov's integral that a soliton under a stationary periodic perturbation yields a spatial chaotic wave pattern. The reduced case in this problem, which may be similarly analysed, corresponds to a stationary forced KdV equation with an additional shift term Cu_X. This means for certain oblique soliton the topography have to be so chosen that its contours are straight and parallel to soliton's. Nevertheless we do not have any analytic methods in general for PDE problems.

6. Conclusions

Three examples of the supercritical strut flow problem are solved by applying three different methods: the generation of oblique multi-soliton wave pattern by the inverse scattering method; the generation of solitary wave train by a wedge-shaped strut by Whitham's average method; and the evolution of an oblique single soliton over a periodically varying topography by the adiabatic approximation method.



(a) $T = 0 - 150$ (b) $T = 100 - 120$

FIGURE 7. Phase portraits in the plane $(\kappa, \dot{\kappa})$.

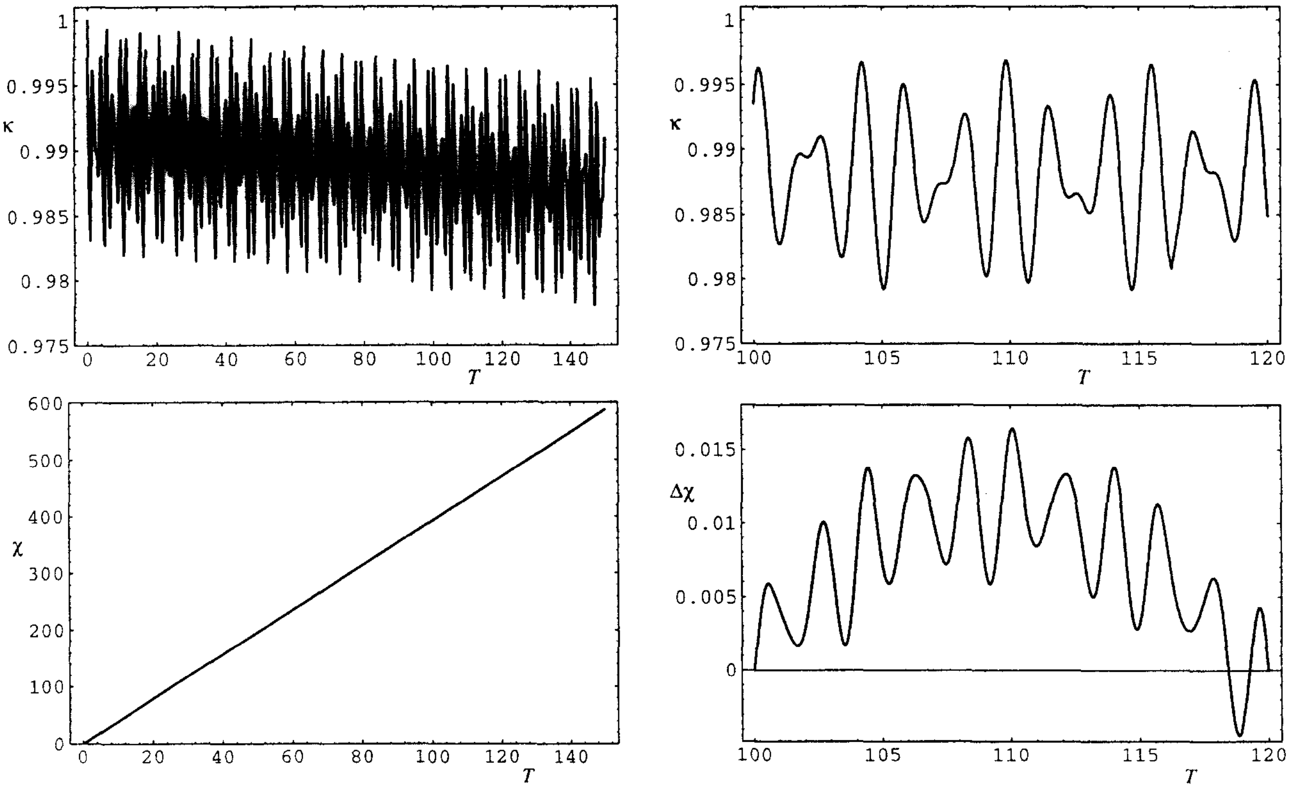


FIGURE 8. Soliton amplitude κ and phase (position) χ or $\Delta\chi$ vs. T.

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